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DOING MATH AS RESEARCHERS DO IT

One of the significant outputs of the MatLan project is the present elective course curriculum, written for high-school math teachers who implement or intend to implement math research workshops for students.

Curriculum for elective course

2016

Project coordinator

Colegiul Național „Emil Racoviță” Cluj-Napoca, Romania

Partner school

Lycée d'Altitude de Briançon, France

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Curriculum for the elective/ optional course

Doing math as researchers do it

Age group: 14 – 19/ high-school students

Number of lessons per week: 1-2

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Cluj-Napoca, 2016

Content

Acknowledgements

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Introduction

MATh.en.JEANS/ MeJ (Méthode d'Apprentissage des Théories mathématiques en Jumelant des Etablissements pour une Approche Nouvelle du Savoir) are workshops for students of different age groups. These workshops encourage the students to engage in and eventually learn math by discovering and researching it. The MeJ workshop develops students’ creativity, initiative, critical thinking, problem solving skills, etc., and gives students the chance to exchange ideas by working in groups both within their MeJ workshop and with students from a different MeJ workshop. MeJ workshops have a long history in France, where they started in 1985, and a very brief history in Romania. In 2013, we began to plan joint MeJ workshops through a bilateral collaboration between our high-schools, which is continued via the Erasmus+ project *Learning math and languages through research and cooperation - MatLan*.

One of the *MatLan* project objectives is to include the MeJ workshop into the school's curricular provision as an elective course. That’s why one of the significant outputs of our project is the present elective course curriculum, written for high-school math teachers who implement or intend to implement math research workshops for students. We are aware that different countries have different educational systems and different requirements related to the curriculum of an elective course. However, this curriculum contains the necessary information for a course curriculum (title, age group, argument, objectives, competences, content, methodological suggestions, assessment and evaluation, resources) which might be easily adapted to national/ local requirements by interested teachers.

The curriculum of this elective course has been developed by considering the elective course curricula designed and used in the two partner schools: Colegiul Național „Emil Racoviță” Cluj-Napoca, Romania (Văcărețu, Programă pentru disciplina opțională Atelier de cercetare matematică/ Studiul matematicii prin cercetare, 2016) and Lycée d'Altitude de Briançon, France (Proal, 2016). The two curricula are, in fact, adaptations of this curricula to Romanian and French high-schools; the two curricula are available on the MatLan project website.

It would be rewarding for us to know that you find this guidebook useful in your work. If you care to leave some feedback, or if you would like to learn more details about our experience with the math research workshop, look us up at http://matlanproject.weebly.com, where you will find contact details and also links to each project partners’ website.

The authors

Argument

The *Doing math as researchers do it* elective course is a scientific and technical workshop for high-school students. There is no special requirement related to the students' math knowledge or math skills and all high-school students who want to take the course should be welcomed. High-school students (aged 14 – 19) may work together within this course. The course allows students to meet researchers and experience an authentic math-research process in school, with both a theoretical and an applied dimension.

This elective course represents:

- a possibility to individually support the student in studying math;
- a new approach to learning math;
- a way to support students' carrier orientation through contacts with professional math researchers;
- a chance to equip high-school students with tertiary education working/ learning techniques.

This elective course encourages the students to engage in and learn math by discovering and researching it. It develops students' creativity, initiative, critical thinking, problem solving skills, etc., and allows them to exchange ideas by working in groups.

In this elective course, mathematics research topics are launched by professional researchers. Small, 2-3-student groups, choose one of the proposed problems and do research work to solve it. The students have to organize their work, identify the resources (strategies, knowledge, experience, equipment, software, materials); decide how the resources will be used for building and maintaining a shared understanding of the task and its solutions. The students' activity is facilitated by the teacher. A professional math researcher should participate in the course and periodically meet with the students in order to discuss the students' research work and the methodology of math/ scientific research.

Photo 1: Students discussing their research results with a professional researcher

This elective course, through its cross-curricular approach, may contribute to the development of some of the key competences for lifelong learning from the European framework (EC, 2006):

- mathematical competence and basic competences in science and technology: by being part of this elective course, the students develop math-specific competences (e.g. modelling mathematically, representing mathematical entities, reasoning mathematically, making use of aids and tools) and experiment an additional way of learning math and informatics;
- digital competence: the presentation of the research topic and results implies the use of digital technology; moreover, the students collect and use information by different means, including ICT;
- learning to learn competence: individually and in groups, students organise their research work; they should be aware of participating in their own training through the organisation of the research work, the reflections in the research journal, presentation of the research results, use of the self-assessment tools. All these skills may/ will be used in other contexts, too (e.g. in tertiary education);
- communication in the mother tongue: students develop oral and written communication skills through lots of activities (e.g. sharing their research work with peers, making oral presentations of the research results, writing the research article).

The regular work on the research topic capitalised on through participation in scientific events allows each student to develop the targeted competences. In this way, both students who struggle in math and those who excel benefit from their participation in the elective course.

Photo 2: Students share results of the research work

Objectives

The objectives of the *Doing math as researchers do it* elective course are:

- to introduce students to an additional way of doing math;
- to show students, by engaging them in math research, that math is wonderful;
- to develop students' curiosity and enjoyment of doing math through a method which fosters autonomy and imagination;
- to individually support students in studying math;
- to introduce students to academics so they would better understand math research careers.

Photo 3: Students collaborate to understand the research topic

Competences

To define the competences developed within this elective course, we first analysed the Researcher Development Framework (Vitae, 2010) and adapted the model to the context of the math-research workshops. We included both transversal 21st-century skills and mathematical competences, as we consider that the workshops provide opportunities for both type of competences to develop. The diagram below shows the competencies potentially developed in the math-research workshops.

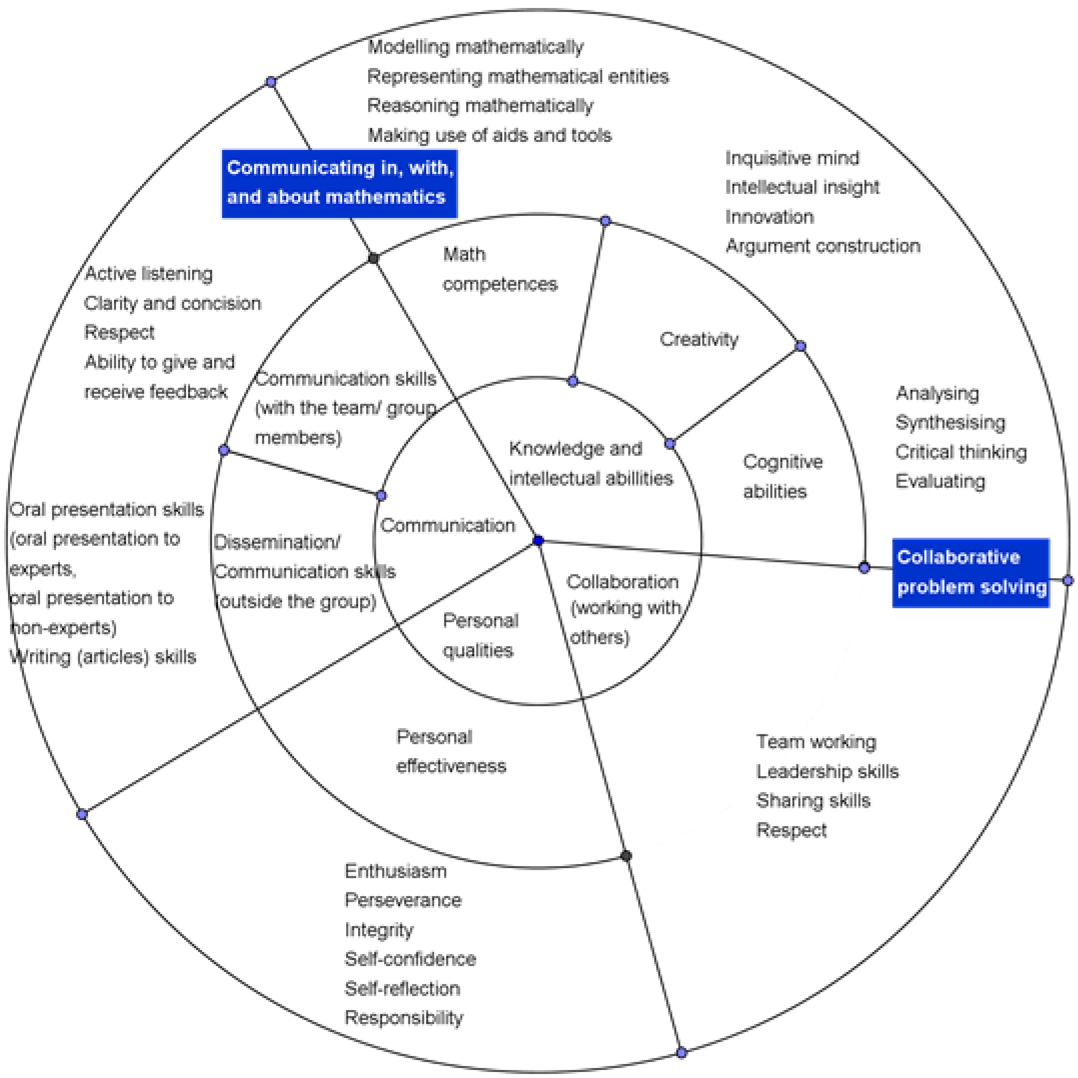

Figure 1: Competences developed within math-research workshops

It is impossible to develop all the competences within one school year, but all the competences might be developed if the students participate in the elective course for 3-4 consecutive years. We recommend focus on the development of 3-4 competences per year. For example, during one school year you may decide to focus on the development of the following competences:

- collaborative problem solving;
- making use of aids and tools (IT included);
- oral presentation skills;
- (article) writing skills.

Collaborative problem solving involves two or more people working towards a joint problem solution. All the participants must contribute their resources, skills, etc. so the problem can be solved. Those two or more people will have a common goal; the resources needed to solve the problem are beyond the capacity of either person alone. If people work together, they might be able to work it out. Collaborative problem solving, or working with others to solve a common challenge, includes the contribution and exchange of ideas, knowledge, and/or resources in order to achieve a shared goal.

This competence brings together two skills: problem solving - which can be included in the category *Knowledge and intellectual abilities*, related to ways of thinking (Blinkley et al., 2012); and collaboration – which, in our model, is a category by itself, related to ways of working (Blinkley et al., 2012).

Making use of aids and tools (IT included) competence has been defined by Mogens Niss and Tomas Hojgaard (Niss, 2011) as consisting of:

a. having knowledge of the existence and properties of the diverse forms of relevant aids and tools used in mathematics and having an insight into their possibilities and limitations in all sorts of contexts;
b. being able to reflectively use such aids and tools.

OECD refers to making use of tools as an ability (OECD, 2013). It is defined as follows: "this ability involves knowing about and being able to make use of various tools that may assist mathematical activity, and knowing about the limitations of such tools." It should be noted that the OECD definition is quite similar to the definition provided by Niss and Hojgaard; *reflection* makes the difference, as the Danish authors mention the reflective use of aids and tools.

To define this competence, it is necessary to first define aids and tools. According to Cambridge Dictionaries (Cambridge University, 2015), *a tool is a piece of equipment that you use with your hands to make or repair something, while an aid is a piece of equipment that helps you to do something*. For example, a protractor is a tool – we can measure angles with it, while Geogebra is an aid – we can build a geometrical configuration with it.

Oral presentation of the math research results implies the presentation in front of a large number of people, not necessarily math experts/ professionals, of the results, research approaches and/ or methods.

The students have to adapt their discourse or presentation to the audience in order to make them understand the math concepts and procedures. Depending on the audience and the nature/ type of communication, the students might decide to use aids and/ or specific tools (computers, scale-models, etc.).

Writing the research article implies the use of both colloquial and scientific/ mathematical language for expressing ideas, for presenting the process and the knowledge gained. When writing the research article, the student will have to comply with rules and standards set by the scientific community.

Content

The content of this elective course should be related to the math research methodology and presentation of the math research results.

Here are some suggestions:

- research – as a way of solving problems;
- techniques and strategies for collaborative math research (stages, relationships among team members, how to collaborate);
- math/ scientific research methodology: formulating hypotheses, designing the experiments/ physical or virtual models, performing the experiments, interpretation of the results, mathematical proof of the research results, formulating the conclusions;
- design and implementation of the research plan;
- physical/ virtual models;
- general characteristics of the audience;
- types of oral presentations of the research results;
- guide to creating different types of oral presentations;
- scientific poster design;
- guide to creating research posters;
- structure of a math/ scientific research article;
- standards and rules for writing math/ scientific research articles.

Methodological suggestions

The research topics

Of major importance in this elective course are the research topics. They are math-research problems which should invite investigation. The research topics should be formulated by professional researchers. The research topic text should be easy-to-read, formulated in everyday language. The research problem should have only one question, it should allow for multiple approaches, and it should be an open problem which may have more than one solution.

Please find below some examples of research topics:

The vaults. We have stones of polygonal shapes (except rectangle) and we have to build a vault between two pillars. A stone is in equilibrium if the perpendicular bisectors of the contact surfaces and the vertical line passing through the centre of gravity of the stone are concurrent. To figure out how to build such a vault, study its shape.

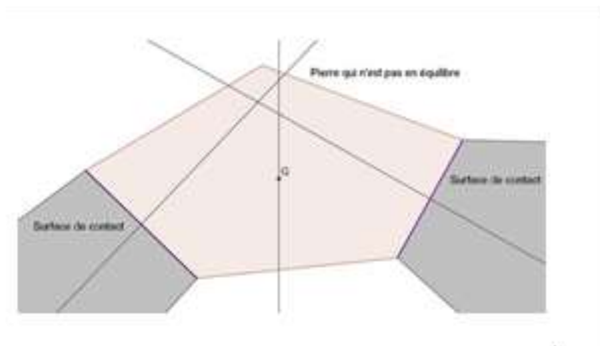

Figure 2: The vaults

The game of Hex. Hex is a board game played on a diamond grid with hexagonal cells. There are two players: a blue one and a red one. Players take turns placing a stone of their colour on a single cell. The goal of the game, for the 'red player', is to form a connected path of red stones linking the opposing sides of the board marked with red, and vice versa for the blue player. Develop a winning strategy.

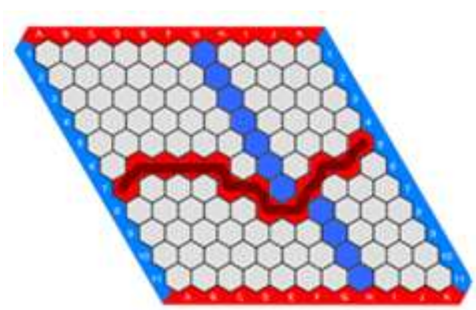

Figure 3: The game of Hex

Location. We must place a minimum number of beams (segments) in a house to be able to know in what room a person is. Each beam is capable of telling us the number of times it has been cut. How to arrange the beams to know where the person is located at any time of the day or night?

Figure 4: Location

Room Monitoring. We must place a minimum of motion sensors in a building knowing that a sensor reacts if a person is in the room where the sensor is or in a nearby room. The activation of the sensors should help to pinpoint in what building room the intruder is.

Figure 5: Room monitoring

Growth of crystal stones. We model the growth of a crystal stone in the following way: starting from a cube (stage 0), let's place the same cube on each of its faces to obtain the “crystal stone” as shown in the figure (step 1). Then let's add cubes to get the "crystal stone" as shown nearby (step 2). Let's keep on doing the same over and over. What can you say of the structure after several steps?

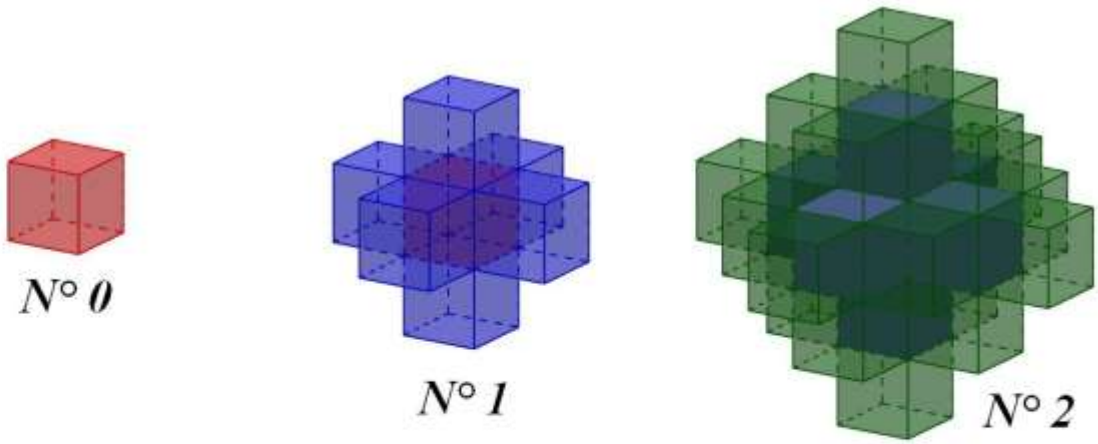

Figure 6: Image related to the research topic Growth of the crystal stone

Organisation of the math research

The inquiry-based learning (IBL) approach is the fundamental approach to this course. IBL is used in a collaborative and interactive context.

In brief, the following stages are part of this elective course: math research topics are launched by professional researchers after being discussed with the teachers; students get a math research topic by choosing what they want to work on; students work in groups for doing the research; teachers and professional researchers facilitate student research; students share their research results in different scientific events; students write and publish a scientific article about their research findings.

We expect students to choose and use what they think are the most effective mathematical concepts, methods and procedures for carrying out the research. It is not **the** solution which is the most important in the context of this elective course, but the process. The students’ work in this elective course **is not** about searching the Internet in order to make a presentation.

The groups of students will take the following steps to solve the problem:

Understand the problem	• Each student explores the problem space[1] and identifies its elements and aspects. • In the group, students collect and share information about the problem elements and how they are linked. In this stage, the group of students defines the problem space collaboratively.
Devise a plan	• In the group, students discuss; they look for patterns and links between elements of the problem, analyse particular cases, organize and categorize the information/ data, both within and across the individual areas of expertise; students re-formulate the problem in a familiar/ mathematical language and plan their approach to solving the problem.
Carry out the plan	• In the group, through discussions and collaboration, the students set up procedures and strategies to solve the problem. They begin to formulate rules or contingencies associated with actions and observations ("if …, then"). These rules or contingencies lead to generalisations.
Look back and check	• Students test hypotheses by challenging generalisations, check process and solutions.

Table 1. Approach to collaborative problem solving

The major stages of the problem-solving process identified by Polya (1973) are appropriate for our research approach: understanding the problem, devising a plan, carrying out the plan, looking back (examining the solution obtained and checking the results) – see Table 1. Both inductive and deductive reasoning are embedded within this process, as the research task challenges the students to detect information, identify patterns or analyse particular situations (as part of the inductive process), and then identify rules and test hypotheses (as part of the deductive process).

Within the research work, collaboration is essential – e.g. in our experience, when first confronted with the research topic, individuals had no idea where to start. After discussing with their peers, ideas about how to start their research emerged. We actually engage the students in their zone of proximal development. The zone of proximal development is the difference between a student's actual developmental stage and his or her potential developmental stage when collaborating with a more able peer or teacher (Vygotsky, 1978) – see figure 7. Students on the elective course teams have different skills,

[1] The problem space is the way an individual or a group of individuals think about the transformation between the beginning stages of the problem and throughout finding a solution.

different knowledge and each of the students can be the ‘knowledgeable other’ in different moments of their work.

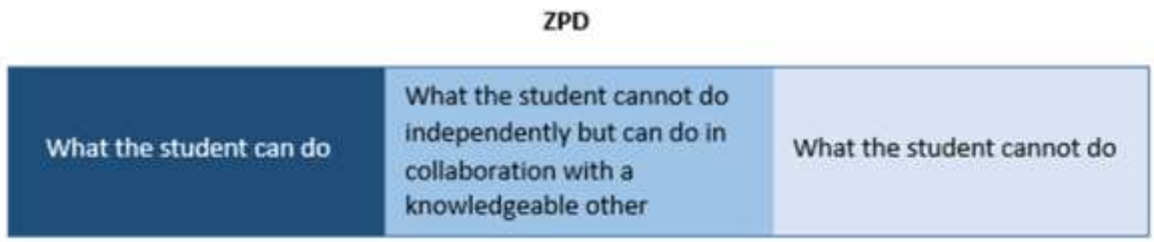

Figure 7. Diagram of the Zone of Proximal Development (ZPD)

Within the math research process, the students might use different aids and tools (e.g. calculator, Mathematica software, Geogebra, Python, plotting paper, Lego bricks, plywood models, mock-ups, etc.) for recognising mathematical structures and/ or portraying mathematical relationships and/ or assist in implementing processes and procedures to determine mathematical solutions.

After finalising the research, the students should present the research process and results to an audience – in the school (e.g. math forums) and outside the school (e.g. students’ conferences/ congresses). In designing and doing the presentation, the students should consider the characteristics of their audience (high-school/ lower secondary students, math teachers, professional researchers, parents, etc.) and the type of the presentation (plenary, interactive presentation). The preparation of the presentation(s) gives an opportunity for each student to become aware of the results achieved (by himself or by his group) and to reorganise the new knowledge related to both the content and the process of the research. To present the research work and results, students need to focus on the main results, order the stages of the research process, and find an appropriate way to reach the audience.

The last stage of the research workshop also includes a learning activity in which writing plays a vital role. The students write their articles, which are sent to be validated by a professional researcher (the professional researcher who launched the research topics). The articles should comply with standards of math/ scientific research articles.

The teacher’s role

During the lessons of this elective course, the teacher should facilitate the learning. The teacher

- creates an environment which is appropriate for research and discussions without giving answers to the students’ questions;
- encourages the students to communicate their thoughts, to rephrase their colleagues’ statements or questions, to allow enough time for their colleagues to talk;

- supports the students to formulate questions, to find appropriate ways to model the problem to communicate the results.

Briefly, the teacher encourages, comforts, invites to discussions, invites to rigorous proof, (sometimes) suggests the tools if the students ask for it, advises in organizational matters and in the presentation of the results. The teacher **should not** transform the research topic into a sequence of questions.

The researcher's role

The researcher should participate in the lessons of this elective course every six weeks or so. The researcher encourages students to formulate questions related to the research topic, advises on scientific research methodology, provides feedback on their work in progress, and validates the findings of the students' research and their research articles.

Tips

- Before starting to implement this elective course, the teacher should find a professional math researcher who accepts to collaborate with the teacher for the implementation of this course. This elective course is a replica of math research done by professional researchers, that's why the professional researcher's input is very important. The professional researcher and the teacher collaborate in proposing the research topics.
- Sometimes, a research topic doesn't lead to interesting solutions – but this shouldn't be an issue. It is not possible to predict how the students will approach the research and how they'll advance within the research work.
- During the school year, students might communicate with the researcher via face-to-face meetings or video-conferences.
- There are many research topics available on the MATh.en.JEANS website.
- If there are two schools (from the same country or form different countries) who implement this elective course, the students from the two schools may work together within a twinning/ e-twinning project. In such a situation, the students from both high-schools research the same topics and they put together the results of the research. The exchanges may be done by in face-to-face meetings, via video-conferences and/ or via e-learning or e-communication platforms. If the twin schools are from different countries, communication in a foreign language competence is developed, too.
- Students' participation in conferences/ scientific events contributes the dimensions of cultural awareness – especially if these events take place in different places – and also career orientation by meeting and discussing with professional researchers.

Assessment and evaluation

Assessment and evaluation are implemented by using different methods (observation, analysis of the information and reflections recorded in the students' research journals (logbooks), self-assessment, analysis of the research articles/ presentations, peer-evaluation, etc.) and focus on the competences which the teacher decides to develop during one school year.

The teacher should adapt existing assessment tools or devise new tools for assessing the students' progress. A possible approach to developing the assessment instruments is the *MatLan* project team approach, described in the Conference paper (Văcărețu, An approach to assessing students' competences, 2015). *To develop the assessment instruments, we started by defining each competence and making it visible. Observing what students do, make, say and write while researching the math topics in the math-research workshop and analysing relevant literature on the respective competence, we identified the skills and sub-skills, as well as the student behaviour which demonstrates mastery of each identified sub-skill. Next, we described three levels of performance (novice, competent, expert) for each behavioural indicator and the developmental progression of the respective competence. Developmental progressions are tools which originally resulted from the work of Piaget (Hurst, n.d.). They are carefully designed detailed 'maps' which illustrate the increasingly sophisticated behaviours that a learner will display as they progress from being a novice to being an expert in any domain of learning (University of Melbourne, 2015). Developmental progressions provide a tool for the teacher to be a systematic observer of student progress. They can help the teacher to notice the kinds of actions and behaviours that students demonstrate and to interpret these observations as evidence of skills and subskills growth. The developmental progression might be used as a frame of reference to interpret what they see the students doing, saying, making, or writing in math research workshops. Finally, we decided on the assessment methods and we designed the assessment tools by using the developmental progression for the respective competence.*

We would like to point out some important issues:

- the assessment tools have to be discussed with the students at the beginning of the school year. They have to understand the tools (especially if they have to use them) and they have to understand what we expect from them – that is why modelling actions and behaviour or provision of examples thereof are a must.
- The student logbook is an important tool for assessing competences developed during the elective course. In the logbook, the students write down what they have done to find solutions to the research topics during the lessons and at home, tasks for the next lessons and their own reflections on what they learned while working on the research topic about themselves, about their peers and new (curricular area) knowledge.
- Assessment tools can be easily developed if we start from the developmental progression.

- An analysis of the especially significant differences between the results of the teacher's observations and student's self-assessment is recommended. Comparison between assessments in different moments of the school year will inform the development of each student's assessed competence within the elective course.

In the annex, you can find some more ideas for shaping the assessment and evaluation of the students' achievements in the elective course.

Annexes

Methods and instruments for assessing students' collaborative problem solving (CPS) competence

For assessing students' collaborative problem solving competence developed within the math-research workshop, we suggest the following methods and instruments:

Methods	Instruments
Observation, analysis of the information and reflections in the students' logbooks (2 times per school year)	The CPS observation sheet
Self – assessment (2 times per school year)	The CPS self-assessment sheet

Table 2. Methods and instruments for assessing students' collaborative problem solving

The students' logbooks are useful only for identifying aspects mentioned in the observation sheet which the teacher didn't/ couldn't observe during the lessons or for checking the accuracy of the observed actions and behaviour. If there is no concordance between the teacher's observations and the student logbook, the teacher might decide to take a closer look at the specific discordant issues. The highest level of progression where all the activities or behaviour are marked will be considered the student's level in the development of the collaborative problem solving competence, while the level on the progression where about half the activities or behaviour are marked is the student's zone of proximal development.

For recording the assessment of the students' collaborative problem solving competence, teachers will use reporting sheet 1 (see below). The reporting sheets are adaptations of the *Learning readiness reports* (University of Melbourne, 2015).

The observation sheet and the self-assessment sheet can be easily developed if we start from the developmental progression. To describe the developmental progression of collaborative problem solving (CPS) within the math research workshop, we adapted to our context the empirically validated developmental progression on one dimension developed within the ATC21S project (ATC21S Project, 2014).

The developmental progression incorporates both social (light orange colour in the table) and cognitive skills (blue colour in the table).

Level	Level description
6	The student works collaboratively through the problem solving process and assumes group responsibility for the success of the research tasks. The student works through the problem efficiently and systematically using only relevant resources. He/ she plans the research strategy based on a generalised understanding of cause and effect, and reorganizes understanding of the problem. The student tailors communication, incorporates contributions and feedback from peers and resolves conflicts.
5	The student identifies the necessary sequence of subtasks; explores more than four particular cases by using ICT (e.g. Geogebra) and/ or manipulatives. The student's actions are planned and purposeful, identifying cause and effect and basing their goals on prior knowledge. He/ she reconstructs the understanding of the problem. The student promotes interaction and responds to peers' contribution but may not resolve differences.
4	The student divides the research task into subtasks, perseveres to successfully complete subtasks and simpler tasks, and explores four particular cases. The student identifies connections and/ or patterns in approaching the research task; he/ she identifies sequences of cause and effect, and modifies hypotheses. The student is aware of his/ her and the peers' abilities. They reach a common understanding, start to plan strategies for finding solutions and refine goals with their peers.
3	The student demonstrates effort towards finding solutions to the research task by stating the research topic in his/ her own words, and exploring three particular cases. He/ she begins to share resources and information with the peers – resources and information shared are sometimes not relevant. The student sometimes asks for support from the teacher. The student reports his/ her activity.
2	The student attempts to better understand the problem through limited analysis; s/he explores two particular cases. He or she sets general goal, begins testing hypotheses. Interaction with peers is limited to brief acknowledgments on significant issues related to the research.
1	The student explores the problem space independently with no evidence of collaboration. His/ her approach is unsystematic and focusing on isolated pieces of information; s/he explores one particular case. No evidence of participation, interaction with peers is limited to brief acknowledgments without providing information or resources.

Table 3. The developmental progression of the CPS competence within the elective course

Reporting sheet 1 - CPS

Student name: ______________________________________ **Date:_______________**

Level	Level description	Teacher assessment	Student assessment
6	The student works collaboratively through the problem solving process and assumes group responsibility for the success of the research tasks. The student works through the problem efficiently and systematically using only relevant resources. He/ she plans the research strategy based on a generalised understanding of cause and effect, and reorganizes understanding of the problem. The student tailors communication, incorporates contributions and feedback from peers and resolves conflicts.		
5	The student identifies the necessary sequence of subtasks; explores more than four particular cases by using ICT (e.g. Geogebra) and/ or manipulatives. The student's actions are planned and purposeful, identifying cause and effect and basing their goals on prior knowledge. He/ she reconstructs the understanding of the problem. The student promotes interaction and responds to peers' contribution but may not resolve differences.		
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Methods and instruments for assessing the students' competence in *making use of aids and tools (MUAT)*

For assessing students' competence in *making use of aids and tools* developed within the elective course, we suggest the following methods and instruments:

Methods	Instruments
Observation (twice per school year) **Analysis of the information and reflections in the student's logbooks (twice per school year)** **Analysis of the student's articles/ oral presentation (once per school year)** **Analysis of the student's reporting sheet (MUAT) (each time the student uses a tool/ aid)**	The MUAT observation sheet

Table 4. Methods and instruments for assessing the students' competence in *making use of aids and tools*

The students' logbook, articles/ oral presentations, and the student reporting sheets (MUAT) are useful only for identifying aspects mentioned in the observation sheet which the teacher didn't/ couldn't observe during the lessons or for checking the accuracy of the observed actions and behaviour. If there is no concordance between the teacher's observations and the student logbook/ article/ oral presentation/ reporting sheet, the teacher might decide to take a closer look at the specific discordant issues and to have a short interview with the student. The highest level of progression where all the activities or behaviour are marked will be considered the student's level in the development of the MUAT competence, while the level on the progression where about half the activities or behaviour are marked is the student's zone of proximal development. For recording their assessment of the students' competence in *making use of aids and tools*, teachers will use reporting sheet 2 (see below).

If the student enrols for the elective course for two (or more) consecutive years, it makes sense to compare results of this assessment across the two (or more) years. We assume that as long as one is doing the math research activity, there can normally be stagnation or growth of the competency.

The developmental progression supports the teacher in carefully observing his/ her students and identifying areas where there is room for improvement.

Level	Level description
6	The student identifies and uses at least 3 familiar or non-familiar, complex aids and tools (with multiple functionalities) which may assist in implementing processes and procedures for determining mathematical solutions. Constructs new tool(s) - necessary for understanding and solving the research task. Interprets the resulting findings from using the tools and aids. Describes the possibilities and the limitations of the used tools by illustrating what s/he means, after reflecting to understand and evaluate the merits and limitations of the tool. Evaluates the use of the aids and tools and communicates their learning from this experience. Reflects on what else s/he could have used, and why s/he ended up not using those aids and tools.
5	The student identifies and uses 3 familiar or non-familiar, complex aids and tools (with multiple functionalities) which may assist in implementing processes and procedures for determining mathematical solutions. Has an idea how to create a new tool or aid necessary for understanding and solving the research task (e.g. a programme). Interprets the resulting findings from using the tools and aids. Describes the possibilities and the limitations of the used tools by illustrating what s/he means, after reflecting to understand and evaluate the merits and limitations of the tool. Evaluates the use of the aids and tools and communicates their learning from this experience. Reflects on what else s/he could have used.
4	The student identifies and uses 3 aids and tools for implementing processes and procedures for determining mathematical solutions. Searches for appropriate tools and aids s/he might use when researching the topic, learns how to use them. Describes the possibilities and the limitations of the used tools by illustrating what s/he means. Evaluates the use of the aids and tools and communicates their learning from this experience.
3	The student identifies and uses at least 2 aids and tools, familiar or non-familiar. Using the tool involves a sequence or process of linking different information. The aids and tools used may assist in implementing processes and procedures to determine mathematical solutions. Describes the possibilities and the limitations of the used tools by illustrating what s/he means. Evaluates the use of the aids and tools.
2	The student identifies and uses 2 tools and aids s/he already used during math lessons. Using the tools involves a sequence of processes. The used aids and tools may assist the student in formulating the research task mathematically. Describes, in general terms, the possibilities and the limitations of the used tools and aids. Analyses the use of both aids and tools (which aids and tools they have used, why they have used them).
1	The student identifies and uses 1 familiar tool and/ or aid (e.g. measuring instruments, objects that are mentioned in the text of the research task). S/he describes, in general terms, the possibilities and the limitations of the used tools. S/he uses the tools and aids in order to recognise mathematical structures in the text of the research topic. S/he names the used tools and formulates one reason for using the identified tools.

Table 5. The developmental progression of the competence *making use of aids and tools* within the elective course context

Reporting sheet 2 - MUAT

Student name: _______________________ Date: ________

Level	Level description	Teacher assessment
6	The student identifies and uses at least 3 familiar or non-familiar, complex aids and tools (with multiple functionalities) which may assist in implementing processes and procedures for determining mathematical solutions. Constructs new tool(s) - necessary for understanding and solving the research task. Interprets the resulting findings from using the tools and aids. Describes the possibilities and the limitations of the used tools by illustrating what s/he means, after reflecting to understand and evaluate the merits and limitations of the tool. Evaluates the use of the aids and tools and communicates their learning from this experience. Reflects on what else s/he could have used, and why s/he ended up not using those aids and tools.	
5	The student identifies and uses 3 familiar or non-familiar, complex aids and tools (with multiple functionalities) which may assist in implementing processes and procedures for determining mathematical solutions. Has the idea to create a new tool or aid necessary for understanding and solving the research task (e.g. a programme). Interprets the resulting findings from using the tools and aids. Describes the possibilities and the limitations of the used tools by illustrating what s/he means, after reflecting to understand and evaluate the merits and limitations of the tool. Evaluates the use of the aids and tools and communicates their learning from this experience. Reflects on what else s/he could have used.	
4	The student identifies and uses 3 aids and tools for implementing processes and procedures for determining mathematical solutions. Searches for appropriate tools and aids s/he might use when researching the topic, learns how to use them. Describes the possibilities and the limitations of the used tools by illustrating what s/he means. Evaluates the use of the aids and tools and communicates their learning from this experience.	
3	The student identifies and uses at least 2 aids and tools, familiar or non-familiar. Using the tool involves a sequence of process or linking different information. The aids and tools used may assist in implementing processes and procedures to determine mathematical solutions. Describes the possibilities and the limitations of the used tools by illustrating what s/he means. Evaluates the use of the aids and tools.	
2	The student identifies and uses 2 tools and aids s/he already used during math lessons. Using the tools involves a sequence of processes. The used aids and tools may assist the student in formulating the research task mathematically. Describes, in general terms, the possibilities and the limitations of the used tools and aids. Analyses the use of both aids and tools (which aids and tools they have used, why they used these aids and tools).	
1	The student identifies and uses 1 familiar tool and/ or aid (e.g. measuring instruments, objects that are mentioned in the text of the research task). S/he describes, in general terms, the possibilities and the limitations of the used tools. S/he uses the tools and aids in order to recognise mathematical structures in the text of the research topic. S/he names the used tools and formulates one reason for using the identified tools.	

Methods and tools to assess the oral presentation and writing (OPW) skills (research results)

The students' oral presentation and writing skills (related to the presentation of the research results) are related to their competence of communication with and about mathematics. The assessment of these skills is done when communication actions take place and when a research poster/ article is produced. As the assessment of these skills relies more on the products (and not so much on the process of preparing the presentations/ articles), it is possible that some of the subskills cannot be assessed during one school year.

For assessing students' oral presentation and writing skills (related to the presentation of the research results), we suggest the following methods and instruments:

Methods	Instruments
Observation (each time the student/ the group of students performs an interactive presentation of the research results, a research poster presentation, a plenary presentation of the research results, when the group of students prepares research articles)	Assessment grids for assessing interactive communication, visual communication, scientific written and oral communication
Self-assessment (each time the student/ the group of students performs an interactive presentation of the research results, a research poster presentation, a plenary presentation of the research results, when the group of students prepares research articles)	Self-assessment sheets for assessing interactive communication, visual communication, scientific written and oral communication
External-evaluation (done by persons from the audience each time the student/ the group of students performs an interactive presentation of the research results, a plenary presentation of the research results)	Public's report cards for assessing interactive communication and scientific oral communication

Table 6. Methods and instruments for assessing the students' oral presentation and writing (OPW) skills (research results)

For recording their assessment of the students' oral presentation and writing skills (research results), the teachers will use reporting sheet 3 (see below).

As these skills are so different, we prefer to provide four developmental progressions corresponding to each of the subskills (interactive communication, visual communication, scientific written and oral communication skills).

Interactive communication (delivered to visitors at a stand, such as in an exhibition/ science fair)	
Level	Description of the level
6	Upon sensing or finding out about the visitors' interests, the student grabs and holds their attention by presenting a captivating and highly relevant part of the research, in a perfectly adjusted language, using the most appropriate tools, engaging in a fluent dialog. For example: involving young students in the analysis of some simple cases, skipping the mathematical proof for an 'unfamiliar with math' audience, pointing out when the audience's questions are similar to his/ her own questions related to the research.
5	Upon sensing the visitors' interests, the student grabs their attention by presenting some of the research, in well-adjusted language, using appropriate tools, which s/he invites the audience to manipulate, play with, etc. He/ she asks and answers questions from the audience.
4	The student provides diverse examples or uses various media to present some specific issues related to the research, engaging with the audience by inviting them to manipulate some materials, and addressing simple questions such as: Do you understand the research topic? What are your ideas about ….? What would be your answer to ….?
3	The student provides diverse examples or uses various media to present some specific issues related to the research, engaging with the audience to a limited extent. For example, the student gives examples, provides a schema, plays games with people from the audience, models/ simulates in order to facilitate the audience's understanding of the research topic. The student uses materials prepared before the interactive communication takes place.
2	The student uses a few materials (paper, computer, etc.) to illustrate some of the points made in the presentation, paying no attention to the visitors' potential interests; fails to engage the visitors – e.g. the student draws a schema on a sheet of paper, uses a computer for presenting some configurations/ collected data/ etc., uses a board game).
1	Paying no attention to the visitors' potential interest, the student delivers an unadjusted presentation of the research topic, and hands visitors a printed outline, failing to engage them.

Table 7. The developmental progression of the interactive communication subskill within the elective course context

Photo 4. Interactive communication (during the Math Youth Forum)

Visual communication	
Level	Description of the level
6	The group has produced and made available online a very clear and attractive item of visual communication (poster in pdf format, very well edited video recording), which shares the research topic, as well as all the major results in an original manner, using highly appropriate computer software/ tools to thoroughly engage viewers. .
5	The group has produced a very clear and attractive item of visual communication which shares the research topic as well as all the major results, using computer software/ tools to show the most relevant parts of it.
4	The group has produced a clear and attractive item of visual communication which shares the research topic as well as most of the results using computer software/ tools to show some parts of it.
3	The group has produced an attractive although not thoroughly clear item of visual communication of the research topic and some results using computer software/ tools to show some parts of it.
2	The group has produced a simple item of visual communication of the research topic and some results, which however tends to be mostly confusing and/or overloaded, blurring the key points to get across.
1	The group of students has produced a simplistic item of visual communication of the research topic (a poster), which fails to attract viewers' interest.

Table 8. The developmental progression of the synthetic and visual communication subskill within the elective course context

Written scientific communication	
Level	Description of the level
6	Seamlessly integrating each members' contribution, the group of students has produced a well-structured, well written and neatly typed research article, which presents the research topic, the approach, the results & the mathematical proof(s) in a highly appropriate language.
5	Incorporating the contribution of each member of the team, the group of students has produced a well-structured, neatly typed research article which presents the research topic, the approach, the results & the mathematical proof(s), in need of limited editing.
4	Some members of the group of students have contributed to producing a neatly typed, quite well-structured and mostly appropriately worded research article, which contains the research topic, their research approach, the research results & the mathematical proof(s).
3	The group of students or a member of the group has produced a complete although poorly structured research article, typed in a rather messy manner, in a register that fails to comply with the standards of a research article.
2	The group of students or a member of the group has typed up an incomplete and poorly structured article, which presents the topic and the research that was carried out, but fails to present the scientific results, etc.
1	The group of students or a member of the group has typed up a simplistic, partial report on the research they have carried out, failing to specify essential information e.g. the topic and/or the approach and/ or results, etc.

Table 9. The developmental progression of the written scientific communication subskill within the elective course context

Oral scientific communication (in a plenary)	
Level	Description of the level
6	The student delivers an oral presentation of his/ her research work using a clear and highly structured visual support (e.g. a slideshow/ PPT). The student provides clear and relevant answers to all the questions from the audience.
5	The student delivers an oral presentation of his/ her research work using a clear and highly structured visual support (e.g. a slideshow/ PPT). The student answers questions from the audience mostly clearly.
4	The student delivers an oral presentation of his/ her research work using a clear well-structured visual support (e.g. a slideshow/ PPT). The student answers some questions, but his/ her answers tend to lack clarity.
3	The student delivers an oral presentation of his/ her research work, using visual support (e.g. a slideshow/ PPT). However, the PowerPoint presentation has too much text and/ or confusing visual effects. He/ she makes the presentation sometimes simply reading the text from the slide(s), and refraining from answering questions.
2	The student delivers a simple oral presentation of his/ her research work, without relying on reading, failing to use visual support to facilitate the understanding of the presentation, and refraining from answering questions.
1	The student presents his/ her research work orally, by simply reading out his/ her notes in a rather dull manner, and refraining from answering questions.

Table 10. The developmental progression of the oral scientific communication subskill

Photo 5: Oral presentation of the research results

Reporting sheet 3 – OPW skills

Name of the student …………

Interactive communication		Assessment		
Level	Description of the level	Audience	Student	Teacher
6	Upon sensing or finding out about the visitors' interests, the student grabs and holds their attention by presenting a captivating and highly relevant part of the research, in a perfectly adjusted language, using the most appropriate tools, engaging in a fluent dialog. For example: involving young students in the analysis of some simple cases, skipping the mathematical proof for an 'unfamiliar with math' audience, pointing out when the audience's questions are similar to his/ her own questions related to the research.			
5	Upon sensing the visitors' interests, the student grabs their attention by presenting some of the research, in well-adjusted language, using appropriate tools, which s/he invites the audience to manipulate, play with, etc. He/ she asks and answers questions from the audience.			
4	The student provides diverse examples or uses various media to present some specific issues related to the research, engaging with the audience by inviting them to manipulate some materials, and addressing simple questions such as: Do you understand the research topic? What are your ideas about ….? What would be your answer to ….?			
3	The student provides diverse examples or uses various media to present some specific issues related to the research, engaging with the audience to a limited extent. For example, the student gives examples, provides a schema, plays games with people from the audience, models/ simulates in order to facilitate the audience's understanding of the research topic. The student uses materials prepared before the interactive communication takes place.			
2	The student uses a few materials (paper, computer, etc.) to illustrate some of the points made in the presentation, paying no attention to the visitors' potential interests; fails to engage the visitors – e.g. the student draws a schema on a sheet of paper, uses a computer for presenting some configurations/ collected data/ etc., uses a board game).			
1	Paying no attention to the visitors' potential interest, the student delivers an unadjusted presentation of the research topic, and hands visitors a printed outline, failing to engage them.			

Synthetic and visual communication		Assessment	
Level	Description of the level	Student	Teacher
6	The group has produced and made available online a very clear and attractive item of visual communication (poster in pdf format, very well edited video recording), which shares the research topic, as well as all the major results in an original manner, using highly appropriate computer software/ tools to thoroughly engage viewers. .		
5	The group has produced a very clear and attractive item of visual communication which shares the research topic as well as all the major results, using computer software/ tools to show the most relevant parts of it.		
4	The group has produced a clear and attractive item of visual communication which shares the research topic as well as most of the results using computer software/ tools to show some parts of it.		
3	The group has produced an attractive although not thoroughly clear item of visual communication of the research topic and some results using computer software/ tools to show some parts of it.		
2	The group has produced a simple item of visual communication of the research topic and some results, which however tends to be mostly confusing and/or overloaded, blurring the key points to get across.		
1	The group of students has produced a simplistic item of visual communication of the research topic (a poster), which fails to attract viewers' interest.		

Written scientific communication		Assessment	
Level	Description of the level	Student	Teacher
6	Seamlessly integrating each members' contribution, the group of students has produced a well-structured, well written and neatly typed research article, which presents the research topic, the approach, the results & the mathematical proof(s) in a highly appropriate language.		
5	Incorporating the contribution of each member of the team, the group of students has produced a well-structured, neatly typed research article which presents the research topic, the approach, the results & the mathematical proof(s), in need of limited editing.		
4	Some members of the group of students have contributed to producing a neatly typed, quite well-structured and mostly appropriately worded research article, which contains the research topic, their research approach, the research results & the mathematical proof(s).		
3	The group of students or a member of the group has produced a complete although poorly structured research article, typed in a rather messy manner, in a register that fails to comply with the standards of a research article.		
2	The group of students or a member of the group has typed up an incomplete and poorly structured article, which presents the topic and the research that was carried out, but fails to present the scientific results, etc.		
1	The group of students or a member of the group has typed up a simplistic, partial report on the research they have carried out, failing to specify essential information e.g. the topic and/or the approach and/ or results, etc.		

Oral scientific communication (in a plenary)		Assessment		
Level	Description of the level	Audience	Student	Teacher
6	The student delivers an oral presentation of his/ her research work using a clear and highly structured visual support (e.g. a slideshow/ PPT). The student provides clear and relevant answers to all the questions from the audience.			
5	The student delivers an oral presentation of his/ her research work using a clear and highly structured visual support (e.g. a slideshow/ PPT). The student answers questions from the audience mostly clearly.			
4	The student delivers an oral presentation of his/ her research work using a clear well-structured visual support (e.g. a slideshow/ PPT). The student answers some questions, but his/ her answers tend to lack clarity.			
3	The student delivers an oral presentation of his/ her research work, using visual support (e.g. a slideshow/ PPT). However, the PowerPoint presentation has too much text and/ or confusing visual effects. He/ she makes the presentation sometimes simply reading the text from the slide(s), and refraining from answering questions.			
2	The student delivers a simple oral presentation of his/ her research work, without relying on reading, failing to use visual support to facilitate the understanding of the presentation, and refraining from answering questions.			
1	The student presents his/ her research work orally, by simply reading out his/ her notes in a rather dull manner, and refraining from answering questions.			

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